Computational Challenges in Flight Control Design

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Abstract

The model based design of advanced flight control systems raises many computational challenges both in developing design adequate models as well as in solving optimization based controller tuning and assessment problems. The generation of uncertainty models for aircraft dynamics and the development of low order aeroelastic models are two modelling problems leading to symbolic and numerical computational challenges. Further challenges arise in solving semi-infinite optimization problems for tuning controller parameters or global optimization problems for controller robustness assessment. The satisfactory solution of such problems represents a **sine qua non** condition for the development of future computer aided design environments for the flight control systems design of the next generation aircraft.

1. Introduction

The main function of a modern flight control system (FCS) of an aircraft is to ensure its safe and economic operation, such that all intended flight missions can be accomplished over a wide range of operating conditions, and unexpected events (e.g. engine failure) can be appropriately handled. The increased safety, and the economical and performance demands led to advanced FCSs of very high level of complexity, involving large development costs. The potential danger that the resulting economical benefits (e.g. low fuel consumption, less weight) are nullified by higher design and maintenance costs requires the development of efficient, cost effective computer aided FCS design technologies, capable to ensure up to 30% reduction of overall development costs of a new aircraft. The reduction of costs by using advanced design methodologies is the central issue for the success of the next generation of transport aircraft.

Although the flight control laws design represents a modest part in the overal FCS design, its role is central for the aircraft operation [1, 2]. For an aircraft, the role of feedback control is well known: it provides tight pilot command tracking, the attenuation of effects of external disturbances (e.g. wind gusts), and improved dynamical behaviour in presence of variations of aircraft parameters (robustness against parametric uncertainties). A common need of many modern feedback control design methodologies is the availability of adequate aircraft dynamical models of different complexities. For instance, accurate nonlinear models are needed for FCS evaluation by simulations. Low order approximate models are necessary for efficient evaluation of various design criteria or constraints. Note that several frequently used flying qualities are formulated on basis of appropriate linear models.

In this paper we discuss two modelling problems: the generation of uncertainty descriptions for aircraft dynamics and the development of low order aeroelastic models. Both lead to difficult computational challenges which are presently not yet satisfactorily solved. Further we discuss the general formulation of flight control laws design as semi-infinite constrained multi-criteria optimization problems and describe an iterative solution method making these very complex optimization problems tractable. Finally, we address the computational aspects of assessment of designed flight controllers. The efficient solutions of some typical assessment problems (e.g. the NP hard stability robustness problem) represent computationally challenging global optimization problems.

Throughout the paper we will often refer to the Design Challenge formulated by the GARTEUR *Action Group on Robust Flight Control* and described in a recent book [2]. The design problem formulation together with the developement of an accurate dynamical model, the *Research Civil Aircraft Model* (RCAM), is described in [3]. The above book contains complete solutions for the GARTEUR robust control design challenge benchmark problem for RCAM. The solution proposed in [4, 5] illustrates the advantages and difficulties of the semi-infinite optimization based design. Post-design stability robustness results for 12 designs for the RCAM benchmark problem are reported in [6].

2. Computational Challenges in Aircraft Dynamics Modelling

The discussion of aircraft dynamics modelling problem in this section is purposely restricted to the development of approximate low order models which are suitable for flight controller synthesis (e.g. to evaluate appropriate flying quality criteria) or in post-design performance robustness assessment of flight controllers. We will not concern with the important details of developing the original aircraft models, focusing mainly on the computational aspects of converting these models in models adequate for various analysis and design tasks. We discuss two aircraft modelling problems, both of them leading to difficult computational challenges.

The first problem is the generation of linearized aircraft models with explicit parametric uncertainty descriptions starting from lumped-parameter nonlinear dynamic models. The parametric uncertain linear models may serve for stability and performance robustness analysis or may be used for robust controller synthesis. The main computational challenges of uncertainty modelling lie in efficient symbolic solution of high order multi-dimensional realization problems and in developing efficient numerical procedures for exact or approximate multi-dimensional order reduction.

The second modelling problem is the determination of lower order approximate aeroelastic models starting from large order high fidelity models resulted from finite element modelling of the flexible aircraft. These models may be used to evaluate criteria for the design of flutter-free flight controllers. The main computational challenge here is the development of efficient computational methods for the order reduction of high order systems (from several thousands to several tenth).

2.1. Parametric Uncertainty Modelling

In many cases, the dynamical behaviour of a flying aircraft can be accurately described by a lumped-parameter nonlinear dynamic model of the form

$$\begin{aligned} \dot{x}(t) &= F(x(t),\, u(t),\, p) \\ y(t) &= G(x(t),\, u(t),\, p) \end{aligned} \qquad (1)$$

where x, u, y are the state-, input- and output–vectors, respectively, and p is a vector of model parameters. The lumped-parameter aircraft model is derived on basis of well established flight physics principles applied to the *rigid-body* aircraft dynamics and results as the interconnection of several dynamical and static subsystems describing different parts of the aircraft dynamics and of the interactions of aircraft with its flight environment. As an example of a lumped-parameter rigid-body aircraft model, we will often refer to the *Research Civil Aircraft Model* (RCAM) [3].

If aeroelastic effects are negligible, then the relatively simple lumped-parameter approximation (1) is completely satisfactory for accurate simulation of the aircraft dynamics, including even real-time simulations. The nonlinear model (1) is also useful for generating linearized models for particular flight conditions and parameter values. Such linear models can be used then for flight control system analysis and design purposes. Since parametric dependencies are explicit, the model (1) can also serve to generate linearized parameter dependent uncertainty models.

The overall goal of uncertainty modelling is to obtain a linear time-invariant state space model with explicit parameter dependencies, which satisfactorily approximates *all* linearizations of the nonlinear model (1) over *all* flight conditions (e.g. all speed/altitude values) and *all* parameter values. Such a linear model may be used for stability and performance robustness analysis as well as in robust controller synthesis.

There are several approaches possible with various degrees of conservativeness to derive uncertain aircraft models [7]. The approach which we describe to obtain a linear parametric representation uses the symbolic linearization of the nonlinear model (1) in a nominal flight condition and generates the uncertainty model using an uncertainty description based on *linear fractional transformation* (LFT). For some nominal values of the model parameters p_{nom}, it is possible to compute numerically an equilibrium point $\{\overline{x}, \overline{u}\}$ of the system (1) by solving the system of nonlinear equations

$$0 = F(\overline{x},\, \overline{u},\, p_{nom}) \;. \qquad (2)$$

Since this is an overdetermined system, usually a so-called "trim condition" is also imposed by specifying some components of $\overline{x}$ and/or $\overline{u}$.

Let $\overline{y} := G(\overline{x},\, \overline{u},\, p_{nom})$ be the corresponding equilibrium value of y. Then by linearization of the nonlinear model (1) in the neighbourhood of an equilibrium point $\{\overline{x}, \overline{u}\}$ we obtain an approximate linearized time-invariant model of the form

$$\begin{aligned} \dot{\widetilde{x}} &= A(p)\widetilde{x} + B(p)\widetilde{u} \\ \widetilde{y} &= C(p)\widetilde{x} + D(p)\widetilde{u} \;, \end{aligned} \qquad (3)$$

where $\widetilde{x} = x - \overline{x}$, $\widetilde{u} = u - \overline{u}$, $\widetilde{y} = y - \overline{y}$, and the system matrices are given by

$$\begin{aligned} A(p) &= \left.\frac{\partial F}{\partial x}\right|_{\substack{x=\overline{x}\\ u=\overline{u}}} , \quad B(p) = \left.\frac{\partial F}{\partial u}\right|_{\substack{x=\overline{x}\\ u=\overline{u}}} , \\ C(p) &= \left.\frac{\partial G}{\partial x}\right|_{\substack{x=\overline{x}\\ u=\overline{u}}} , \quad D(p) = \left.\frac{\partial G}{\partial u}\right|_{\substack{x=\overline{x}\\ u=\overline{u}}} . \end{aligned} \qquad (4)$$

We can freely assume that any non-rational parametric expressions in the elements of the state space model matrices can be replaced by polynomial or rational approximations. Thus the model (3) is a linearized system in a *rationally dependent parametric representation*, with the matrices $A(p)$, $B(p)$, $C(p)$ and $D(p)$ having only entries which are rational functions of the physical parameters $p_1, \ldots, p_q$. To account for dependencies of the entries of the system matrices on particular trim conditions, it is possible to enlarge the parameter vector p by adding some components of the equilibrium point vectors to it.

Obtaining a unique parametric linear model of the form (3), which covers all flight conditions and all possible parameter variations, is obviously a difficult model building task since generally, the elements of the system matrices depend on the equilibrium point where the linearization has been performed, which on its turn depends on the used nominal parameter values. A possible way to overcome this situation is described in [7] in case of RCAM. This involved additional parameter fitting for some entries of system matrices to cover approximately all equilibrium points. To compensate for large nonzero residuals, additional fictive uncertainty parameters can be added to p. Without such enhancements, the resulting linear parametric model (3) provides an accurate description of joint parametric dependencies in the aircraft model only in the neighbourhood of an equilibrium point. Thus, in general, it is questionable if the parametric linear model (3) is appropriate to be used in a μ-analysis and synthesis methodology [8] if large parametric variations are involved, or if the controller to be designed must be robust in all flight conditions. Nevertheless, in such cases an obvious approach seems to be to build several parametric linear models of form (3) in a *representative* set of equilibrium points of the flight envelope for a given aircraft. Clearly, this approach is computationally very involved since it is not clear from the begining which and how many points are necessary to choose to cover the whole flight envelope.

In what follows we sketch very shortly the approach to convert a parametric description of a linear system of the form (3) into an LFT-based uncertainty description. Any uncertainty in a parameter p_i expressed as $p_i \in [\underline{p}_i,\ \overline{p}_i]$ can be transcribed in a normalized form $p_i = p_{i0} + s_{i0}\delta_i$ with $|\delta_i| \leq 1$, $p_{i0} = (\underline{p}_i + \overline{p}_i)/2$ and $s_{i0} = (\overline{p}_i - \underline{p}_i)/2$. This local parameter uncertainty is then expressed as an elementary upper LFT

$$p_i = \mathcal{F}_u\left(\begin{bmatrix} 0 & s_{i0} \\ 1 & p_{i0} \end{bmatrix}, \delta_i\right). \tag{5}$$

Recall that for a partitioned matrix M

$$M = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \in \mathbb{R}^{(r_1+r_2)\times(q_1+q_2)}$$

and for $\Delta \in \mathbb{R}^{q_1 \times r_1}$, the *upper* LFT $\mathcal{F}_u(M, \Delta)$ is defined by the feedback-like formula

$$\mathcal{F}_u(M, \Delta) = M_{22} + M_{21}(I - \Delta M_{11})^{-1}\Delta M_{12}.$$

Since all elements of matrices A, B, C and D are rational functions in parameters p_i, $i = 1, \ldots, q$, the structured parametric uncertainties at the components level can be transformed into structured parametric uncertainties at the level of system matrices by using the properties of LFTs [9]. Specifically, products, sums, divisions of individual variables, can be directly represented by series, parallel or feedback couplings of the corresponding elementary LFTs of form (5). The same is true for the corresponding operations with more general rationally dependent parametric expressions. Finally, elementary matrix constructs like row and column concatenations, or diagonal stacking are immediately expressible by equivalent LFT constructs. Thus, for all system matrices, LFT uncertainty models can be readily generated using elementary LFT operations. Equivalently, if we write the state space description (3) in the form

$$\begin{bmatrix} \dot{\widetilde{x}} \\ \widetilde{y} \end{bmatrix} = S(p)\begin{bmatrix} \widetilde{x} \\ \widetilde{u} \end{bmatrix} = \begin{bmatrix} A(p) & B(p) \\ C(p) & D(p) \end{bmatrix}\begin{bmatrix} \widetilde{x} \\ \widetilde{u} \end{bmatrix}$$

we can express $S(p)$ as an LFT

$$S(p) = \mathcal{F}_u\left(\begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix}, \Delta\right), \tag{6}$$

where the diagonal matrix

$$\Delta = diag[\,\delta_1 I_{n_1}, \delta_2 I_{n_2}, \cdots, \delta_q I_{n_q}\,]$$

has on its diagonal the normalized uncertainty parameters δ_1, δ_2, $\ldots$, δ_q. Note that

$$S_{22} = \begin{bmatrix} A_0 & B_0 \\ C_0 & D_0 \end{bmatrix},$$

where A_0, B_0, C_0 and D_0 are the *nominal* values of the system matrices (for all δ_i set to zero). The order of the LFT uncertainty description of the system (3) is $n_\Delta = \sum_{i=1}^{q} n_i$, where n_i is the order of the block in Δ corresponding to the uncertain parameter p_i.

A systematic procedure to generate LFT-based uncertainty descriptions has been proposed in [10]. Using this approach, three uncertainty models have been generated for RCAM starting from a parametric linear model developed in [7]. This linear model corresponds to a symmetric horizontal flight at constant air speed $V_A = 80\,m/s$. The considered uncertain parameters are the mass m, two coordinates of the center of gravity X_{cg} and Z_{cg}, and the air speed V_A. In the first LFT model the air speed value was assumed exactly known (no uncertainty on air speed). This model served mainly for the post-design assessment of the stability robustness of the designed controllers for RCAM [6]. More involved LFT descriptions have been obtained by adding the air speed as an uncertain parameter. The second LFT model was obtained by taking $C_w = \frac{mg}{\frac{1}{2}\rho V_A^2 S}$ as uncertain parameter instead of m, where ρ is the air density and S is the wing area. The advantage of this model is its lower order in comparison with the third LFT model, where m and V_A are independent uncertain parameters.

The resulting orders of the blocks in Δ for each uncertain parameter of the three LFT models are summarized in the following table. These orders have been obtained by performing repeatedly 1-D minimal realizations on subsystems formed by isolating individual blocks corresponding to a single parameter as suggested in [10]. This order reduction procedure was quit effective, taking into account that the order of initial realizations were sometimes twice as large as the values given bellow.

Parameters	m	C_W	X_{cg}	Z_{cg}	V_A	n_Δ
Model I	17	0	15	3	0	35
Model II	0	43	19	5	23	90
Model III	47	0	30	7	109	193

The resulting LFT-based parametric descriptions are generally non-minimal. The construction of minimal order descriptions, that is, with lowest possible n_Δ, is essentially a multi-dimensional minimal realization problem, for which presently no numerically efficient and reliable procedures exist. Therefore, a special concern has been devoted in the last years to minimize the orders of the initially generated LFTs. One possible approach to generate low order LFTs which are almost always of minimal order for single rational functions is to use "optimal" algorithms with minimum number of operations (additions and multiplications) to evaluate the numerator and denominator multivariate polynomials [11, 12]. Additional heuristics at matrix level [13], [14], [15], like exploiting common expressions in the underlying system matrices, allow to arrive at factored or additively decomposed expressions which often lead to lower order LFTs. This way is certainly effective, because the source of non-minimality in the generated LFTs is almost exclusively of structural nature, that is, it is a result of the existence of common expressions entering in different entries of the underlying system matrices. Therefore, a further possibility to reduce symbolically the order of generated LFTs is by employing graph-theoretic results to eliminate structurally uncontrollable/unobservable parts along the lines of results for 1-D systems [16, 17, 18, 19]. Recent results on tackling with the above aspects applied to RCAM and recently developed symbolic manipulation software in Maple for automatic generation of low order LFT models are described in [20].

Symbolic computations for order reduction of LFTs can only be employed for relatively small order LFTs, since the combinatorial aspects in association with high symbolic manipulation costs prevent their use for large scale LFTs. Alternative approaches for order reduction are based on numerical computations. For a recent survey of existent results see [21]. A possible numerical approach is the exact model reduction techniques for LFT systems as proposed in [22], [23]. This procedure extends the exact model reduction approach for 1-D discrete systems based on balanced truncation to n-D systems. From computational point of view it involves the seeking of minimum rank non-negative definite block diagonal solutions of two Lyapunov-type *linear matrix inequalities* (LMIs). The computed solutions play the roles of Gramians in standard 1-D model reduction (see next section) and are used to construct appropriate truncation matrices to compute the matrices of the lower order LFT-model. It is presently questionable if this approach can be turned into an efficient procedure to derive minimal realizations of LFT descriptions, since no efficient procedures exist to find singular structured solutions of large LMIs. In fact, it turns out that taking this aspect explicitly into consideration, as for example, by combining the solution of LMIs with trace minimization techniques, leads to non-convex optimization problems [21]. Of special importance however is the fact that this approach can be employed to generate lower order LFT-approximations by using balancing related multi-dimensional truncation techniques [23].

An effective alternative approach for exact order reduction of high order LFTs is to use block-diagonal similarity transformation matrices of the form $T = \mathrm{diag}\,[\,T_1, \ldots, T_q\,]$, which commute with the uncertainty structure of Δ (i.e. $T\Delta = \Delta T$), to remove uncontrollable/unobservable parts. Recently, a minimal realization procedure based on appropriate controllability/observability forms has been proposed in [24]. This approach appears to be well suited to be turned into a reliable numerical procedure. However, as already mentioned before, significant reductions of order can be often achieved by using standard 1-D reduction techniques based on orthogonal controllability/observability canonical forms computed separately for each block of Δ [10]. One additional particular aspect worth to be mentioned in this context is the possibility to exploit that frequently the S_{11} matrix in the resulting LFT description (6) has most of its eigenvalues equal to zero (all eigenvalues are equal to zero if $S(p)$ is a polynomial matrix in p). Thus, many of these eigenvalues are uncontrollable or unobservable and therefore a special order reduction procedure can be devised to remove in a first step only those uncontrollable/unobservable eigenvalues which are zero (see [20] for more details).

In summary, the main computational challenges for uncertainty modelling are:

- symbolic generation of low order LFT representations for a complete set of uncertain aircraft parameters (about 20)
- developing efficient numerical algorithms for exact or approximate order reduction of large order LFT models (orders about 200-300).

Both above aspects are very important to arrive to low order LFT models to be further used for the efficient evaluation of criteria for optimization based controller tuning and assessment.

2.2. Low Order Aeroelastic Modelling

The development of accurate aircraft dynamics models is a very complex multidisciplinary task involving a multitude of distinct modelling activities. High fidelity models of flying aircraft can be derived from nonlinear Navier-Stokes partial differential equations describing the complex physics of flows interacting with the aircraft body. Computational fluid dynamics (CFD) is the main tool to perform simulations of such models. Usually, quite expensive wind tunnel experiments complements the theoretical modelling by providing values of unknown model parameters, as for instance various force and moment coefficients or control-surface effectiveness data. The overall simulation model includes also subsystem models for actuators, sensors, various filters, propulsion.

A high fidelity aircraft model describes the complete aerodynamical behavior of the aircraft, including also structure dependent aeroelastic effects. However, the complexity of CFD-based models is tremendous. Finite-element based discretization lead to models with up to 400,000 degrees of freedom, which are clearly beyond the usability for flight controller design purposes. With various simplifying assumptions (no viscosity, no friction in flows, etc.), models of lower complexity (with up to several thousands degrees of freedom) can be obtained which are well suited to account for most of aeroelastic effects. Such effects have significant contributions to the overall dynamics of some modern aircraft to be developed in the near future, as for instance, large civil transport aircraft or aircraft with reduced structural weights but with potential for static instability. Note that the resulting aeroelastic models are usually large order linearized models integrating both rigid-mode and elastic degrees of freedom. For details on high precision aircraft modelling see for example [25, 26, 27].

The use of high-authority feedback control systems leads often to a reduced frequency separation between the rigid-body modes and the elastic modes of the controlled aircraft. The main concern in designing flight controllers for elastic aircraft is the avoidance of *flutter*, a self-excited and often destructive oscillation resulting from improper feedback design. Design criteria expressing flutter-free (e.g. non-oscillating, stable) behavior of the closed-loop aircraft are currently included among standard requirements for flight controller tuning. To evaluate such criteria, eigenvalue computations are usually performed. Thus to allow a cheap evaluation of aeroelastic design criteria the need for accurate low order aeroelastic models (orders up to several tenth) arise in early design phases. In what follows, we discuss some of recent developments in the model reduction techniques applicable for the reduction of very large order linear systems, as for instance aeroelastic aircraft models.

We assume that the original linearized system is given in a state space form

$$\begin{array}{rcl} \dot{x} & = & Ax + Bu \\ y & = & Cx + Du \end{array}, \qquad (7)$$

where x is the n-dimensional state vector, u is the m-dimensional input vector and y is the p-dimensional output vector. For convenience we denote the system (7) with the *$p\times m$ transfer-function matrix* (TFM)

$$G(s) = C(sI - A)^{-1}B + D \qquad (8)$$

as $G := (A, B, C, D)$. Let $G_r := (A_r, B_r, C_r, D_r)$ be an r-th order approximation of the original model ($r < n$), with the TFM $G_r = C_r(sI - A_r)^{-1}B_r + D_r$. A large class of model reduction methods can be interpreted as performing a similarity transformation Z yielding

$$\left[\begin{array}{c|c} Z^{-1}AZ & Z^{-1}B \\ \hline CZ & D \end{array}\right] := \left[\begin{array}{cc|c} A_{11} & A_{12} & B_1 \\ A_{21} & A_{22} & B_2 \\ \hline C_1 & C_2 & D \end{array}\right], \qquad (9)$$

and defining the reduced model G_r as the leading diagonal system (A_{11}, B_1, C_1, D). When writing $Z := [\,T\ U\,]$ and $Z^{-1} := [\,L^T\ V^T\,]^T$, then $\Pi = TL$ is a projector on T along L and $LT = I_r$. Thus the reduced system G_r is given by

$$(A_r, B_r, C_r, D_r) = (LAT,\ LB,\ CT,\ D). \qquad (10)$$

Partitioned forms as above can be used to construct a so-called *residualized (or singular perturbation) approximation* (SPA). The matrices of the reduced model in this case are given by

$$\begin{array}{rcl} A_r & = & A_{11} - A_{12}A_{22}^{-1}A_{21}, \\ B_r & = & B_1 - A_{12}A_{22}^{-1}B_2, \\ C_r & = & C_1 - C_2A_{22}^{-1}A_{21}, \\ D_r & = & D - C_2A_{22}^{-1}B_2. \end{array} \qquad (11)$$

For the SPA (11) we have $G(0) = G_r(0)$. Thus, SPAs preserve the DC-gains of stable original systems.

An important class of methods are based on the modal approach proposed initially by Davison [28] and extended with several many new variants. [29, 30, 31]. The use of the modal approach to reduce aeroelastic models has been discussed in [27, 32]. The importance of the modal approach as a useful model reduction technique resides in its applicability to reduce very large order systems as those arising from aeroelastic modelling. The method can also handle models with lightly damped modes and even unstable systems. Note that in case of very large order systems, the modal technique is one of the very few applicable methods.

The basic formulation of the modal approach is a particular case of the general transformation method (9), where $A_{12} = 0$ and $A_{21} = 0$. The eigenvalues of A are separated according to a certain *modal dominance* criterion. The leading diagonal matrix A_{11} contains the *dominant modes* which are retained in the reduced model, while the trailing block A_{22} contains the *non-dominant modes* which are deleted from the model. The residualization formulas (11) can also be used in conjunction with the modal approach.

The critical computation in the modal approach is the determination of the transformation matrix Z to achieve the modal separation. Fortunately, in the case of aeroelastic models, the state matrix A of the original description (7) has often an almost separated block-diagonal structure, thus to perform the modal separation basically means to reorder small diagonal blocks (most of them 2×2) according to a modal dominance criterion. Several dominance criteria can be used simultaneously, as for example, separation of modes with respect to the expected crossover frequency and/or with respect to their contribution to the input-output system dynamics (see for instance [29]).

In spite of its ability to reduce significantly large order aeroelastic models, the modal approach has some important limitations which prevent its usability as a general purpose method. One aspect is the lack of a generally applicable modal dominance analysis method. For example, systems with many clustered (nearly equal) eigenvalues represent difficult cases for modal separation on basis of simple criteria as those used in [29, 30]. To handle such cases, an enhanced modal approach based on a new dominance measure has been proposed in [31]. Another weakness of the modal approach is the lack of a guaranteed bound for the approximation error. This has as consequence the frequent need to experiment with different approximations on a trial and error basis. Thus, we see the main role of the modal approach in the context of reducing high order aeroelastic models to serve for a preliminary order reduction, achieving reductions of the model order from several thousands to several hundreds. For further reduction more powerful methods with guaranteed bounds are necessary.

The use of the *balanced truncation approximation* (BTA) method [33] to reduce aeroelastic models has been discussed in [32], where several pros and cons are formulated. The main strength of balancing related approaches lies in the guaranteed error bound for the approximation error and the guaranteed stability of the reduced system. The approximation error bound can be computed on basis of the *Hankel-singular values* of the system $G = (A, B, C, D)$, defined as [34]

$$\sigma_i = [\lambda_i(PQ)]^{\frac{1}{2}}, \quad i = 1, \dots n,$$

where P and Q are the controllability and observability Gramians, respectively, satisfying the Lyapunov equations

$$\begin{array}{rcl} AP + PA^T + BB^T & = & 0 \\ A^TQ + QA + C^TC & = & 0. \end{array} \tag{12}$$

If we assume σ_i, $i = 1, \dots n$ decreasingly ordered

$$\sigma_1 \geq \dots \geq \sigma_r > \sigma_{r+1} \geq \dots \geq \sigma_n \geq 0,$$

then the H_∞-norm of the approximation error for an r-th order BTA $G_r = (A_r, B_r, C_r, D)$ is bounded as follows [34]

$$\|G - G_r\|_\infty \leq 2 \sum_{i=r+1}^{n} \sigma_i.$$

The computation of the matrices of the reduced system requires only the determination of the truncation matrices L and T in (10), and not of the full matrix Z in (9). These matrices can be determined using the Cholesky factors of the non-negative definite Gramians $P = S^TS$ and $Q = R^TR$. From the singular value decomposition

$$SR^T = \left[\begin{array}{cc} U_1 & U_2 \end{array} \right] \text{diag}(\Sigma_1, \Sigma_2) \left[\begin{array}{cc} V_1 & V_2 \end{array} \right]^T$$

where

$$\Sigma_1 = \text{diag}(\sigma_1, \dots, \sigma_r), \quad \Sigma_2 = \text{diag}(\sigma_{r+1}, \dots, \sigma_n),$$

the truncation matrices can be determined as [35]

$$L = \Sigma_1^{-1/2} V_1^T R, \qquad T = S^T U_1 \Sigma_1^{-1/2}.$$

Since standard techniques to solve Lyapunov equations do not in general exploit the usual sparseness of A, the solution of the Lyapunov equations (12) for large n, say $n > 200$ rise prohibitive storage requirements $O(n^2)$ and a tremendous computational burden $O(n^3)$. One way to try to overcome this situation is to determine by iterative techniques low rank approximations to the Gramians.

If $P_r = X_r \widetilde{P}_r X_r^T$ and $Q_r = Y_r \widetilde{Q}_r Y_r^T$ are rank r approximate solutions of the Lyapunov equations (12), where X_r and Y_r are $n \times r$ matrices, and $\widetilde{P}_r$ and $\widetilde{Q}_r$ are $r \times r$ positive definite matrices, then the approximate Gramians can be expressed as $P_r = S_r^T S_r$, $Q_r = R_r^T R_r$ with $S_r = \widetilde{S}_r X_r^T$ and $R_r = \widetilde{R}_r Y_r^T$, where $\widetilde{S}_r$, $\widetilde{R}_r$ are Cholesky factors satisfying $\widetilde{P}_r = \widetilde{S}_r^T \widetilde{S}_r$ and $\widetilde{Q}_r = \widetilde{R}_r^T \widetilde{R}_r$. From the singular value decomposition

$$S_r R_r^T = U_1 \Sigma_1 V_1^T$$

the expressions of the truncation matrices result as

$$L = \Sigma_1^{-1/2} V_1^T \widetilde{R}_r Y_r^T, \qquad T = X_r \widetilde{S}_r^T U_1 \Sigma_1^{-1/2}.$$

The computational complexity of this approach is $O(nr^2)$ and the additional memory requirement is $O(nr)$.

Although the above approach does not guarantee theoretically the stability of the resulting reduced order approximation, in some practical applications (see for example [36]) the use of this method in conjunction with Krylov subspace techniques led to useful low order stable approximations. The computation of low order approximations of the Gramians relies on well established iterative methods. The matrices X_r and Y_r result as orthogonal bases for certain r-dimensional subspaces of the Krylov subspaces $\mathcal{K}_k(A, B)$ and $\mathcal{K}_{k'}(A^T, C^T)$, respectively, where

$$\mathcal{K}_k(A, B) := \operatorname{Im}[\, B, AB, \ldots, A^{k-1}B \,]$$

and $r \leq \min\{km, k'p\}$. For instance, the computation of X_r can be performed by using the iterative Arnoldi process (if $m = 1$) or block-Arnoldi process (if $m > 1$). The stopping criteria for the Arnoldi iterations are related to the achievable residual of the corresponding Lyapunov equation. The computation of the reduced order matrix $\widetilde{P}_r$ can be done according to the Galerkin orthogonality condition, by solving an rth order dense Lyapunov equation, or using the *general minimum residual* (GMRES) condition, in which case, a nonstandard matrix linear equation must be solved [37]. Recent enhancements of the GMRES approach [38] allow to cope with the stability preservation requirement of the reduced order model. However, the enhanced approach requires the solution of a 2-blocks minimum distance problem [39] involving the solution of a dense Riccati equation of order r.

Along the same lines, methods have been proposed for single-input single-output systems using different iterative schemes to determine the basis matrices X_r and Y_r for the underlying Krylov spaces. By using appropriate restart techniques for the Lanczos method, stable reduced models can be generated [41]. A similar technique has been employed for the Arnoldi method in [42].

Recently, a generalization of Smith's iterative method led to a different algorithm to compute low rank non-negative definite approximations of the solutions of stable Lyapunov equations [40]. This method allows to determine the Gramians directly in the factorized form $P_r = X_r X_r^T$ and $Q_r = Y_r Y_r^T$, where X_r and Y_r are $n \times r_1$ and $n \times r_2$ matrices, respectively, with $r \leq \min\{r_1, r_2\}$. From the singular value decomposition

$$X_r^T Y_r = \begin{bmatrix} U_1 & U_2 \end{bmatrix} \operatorname{diag}(\Sigma_1, \Sigma_2) \begin{bmatrix} V_1 & V_2 \end{bmatrix}^T$$

where Σ_1 is $r \times r$, the truncation matrices can be determined as

$$L = \Sigma_1^{-1/2} V_1^T Y_r^T, \qquad T = X_r U_1 \Sigma_1^{-1/2}.$$

In summary, the model reduction of large scale systems like aeroelastic models raises several important computational challenges:

- solution of large order sparse or dense model reduction problems using the standard approach (involving the solution of large Lyapunov equations of order up to several hundreds)
- development of approximation techniques for very large order (up to several thousands) sparse model reduction methods (e.g. using Krylov subspace techniques) which guarantee the stability of reduced models.

Although several research groups are actively working on different aspects of these problems, unfortunately at present no completely satisfactory model reduction approach exists for large order systems.

3. Computational Challenges in Robust Flight Controllers Design

The tuning problem of a robust flight control law can be always formulated as a multi-criteria optimization problem to minimize simultaneously several criteria expressing design specifications (or constraints) and FCS performance requirements. Performance robustness requirements impose additionally the satisfaction of all requirements in the presence of arbitrary variations of some uncertain parameters. The control law for a specific flight mission (e.g. longitudinal flight) can be usually parametrized by a set of free parameters, called *tuners*, grouped in a vector T. If the uncertain model parameters are grouped in a vector p, then each design criterion c_i, for $i = 1, 2, \ldots, r$, is generally a function of both T and p. The simultaneous minimization of several criteria can be formulated as a "zero" goal-attainment problem [43]

$$\begin{array}{ll} \underset{\gamma, T}{\text{minimize}} & \gamma \\ \text{such that} & c_i(T,p) - w_i\gamma \leq 0, \; i = 1, \ldots, r, \end{array} \tag{13}$$

where $w = [\, w_1, \ldots, w_r \,]$ is a weighting coefficients vector characterising the relative trade-offs between the criteria. The term $w_i\gamma$ introduces an element of *slackness* into the problem, which otherwise imposes that the goals be rigidly met. Thus, hard constraints can be easily incorporated by setting a particular weighting coefficient to zero (i.e. $w_i = 0$).

Since the criteria in (13) depend continuously on the parameters in p, this problem is generally equivalent to a nonsmooth semi-infinite optimization problem

$$\begin{array}{ll} \underset{\gamma, T}{\text{minimize}} & \gamma \\ \text{such that} & \widetilde{c}_i(T) - w_i\gamma \leq 0, \; i = 1, \ldots, r, \end{array} \qquad (14)$$

where $\widetilde{c}_i(T) := \max_{p\in\mathcal{P}}\{c_i(T,p)\}$, with $\mathcal{P}$ usually being a compact multidimensional interval of $\mathbb{R}^q$.

Since each evaluation of a criterion represents itself a multi-parameter global optimization problem with respect to the uncertain parameters, the solution of semi-infinite optimization problems is a notorously difficult and, in case of many uncertain parameters, a computationally very involved problem [44, 45]. Standard techniques transform the semi-infinite constraints in a large set of constraints on a grid of parameter values. In a more sophisticated setting, a sequence of successively refined grids can be used. To evaluate criteria values in intermediary points, multi-dimensional interpolation techniques can be used. The evaluation of some constraints in a typical FCS design like overshoot, rise time, damping, etc. involve at least a simulation and an eigenvalue computation. Thus, even when a rough approximation of the solution is computed by optimizing on a coarse parameter grid, the overall problem (14) becomes intractable for more than three-four continuously varying parameters. An exhaustive search in the parameter space $\mathcal{P}$ is even less efficient when fine grids are used to convert the continuous parameter variations into a very large set of constraints in predefined grid points.

For the solution of many practical FCS tuning problems a much simpler approach can be usually employed. Here we try to justify, from the perspective of semi-infinite optimization, the heuristic multi-criteria approach used in [5] for the design of a robust autopilot for the automatic landing of a mordern cargo aircraft formulated as a robust control benchmark problem for RCAM [3]. A similar approach has been also described in [46] in a different context. Both of these approaches can be formalized as systematic optimization methodologies which, in many concrete applications, are able to guarantee satisfactory solutions.

For FCS tuning problems, the worst-case parameter combinations usually occur at vertex points in the parameter space $\mathcal{P}$ or at the extreme points of the flight envelope. Thus, it is usually possible to convert the semi-infinite optimization problem over $\mathcal{P}$ into a finite dimensional optimization by restricting drastically the infinite parameter space to a representative finite set $\mathcal{P}_{fin} \subset \mathcal{P}$ of a few worst-case parameter combinations and/or flight conditions. This allows to replace the semi-infinite constraints in (14) by a finite number of constraints defined by the points in $\mathcal{P}_{fin}$. With this replacement we obtain a finite dimensional goal-attainment problem

$$\begin{array}{ll} \underset{\gamma, T}{\text{minimize}} & \gamma \\ \text{such that} & c_i(T,p) - w_i\gamma \leq 0, \; i = 1, \ldots, r, p \in \mathcal{P}_{fin}. \end{array} \qquad (15)$$

By solving the goal-attainment problem (15), we get the best solution T^* minimizing simultaneously all criteria over the finite parameter set $\mathcal{P}_{fin}$. Note that, T^* is in general a nonunique Pareto-optimal (or noninferior) solution. For such a solution any improvement in one criterion leads automatically to a deterioration in another criterion. For the solution of (15), standard numerical methods for solving nonlinearly constrained optimization problems can be used. Well suited as solution method is SQP – the sequential quadratic programming method [45] (see also [47] for an overview of SQP).

The solution of the overall goal-attainment problem (13) in the context of FCS design can be frequently computed with a good approximation by using the following iterative procedure:

0. Select an initial representative finite set $\mathcal{P}_{fin}$.
1. Solve the finite dimensional problem (15) to determine the optimal tuner T^*.
2. For the computed T^*, perform a worst-case parameter study for all constraints over a coarse grid on $\mathcal{P}$ to select a set of worst-case parameter combinations, $\mathcal{P}_{worst}$.
3. If all constraints are satisfied in $\mathcal{P}_{worst}$, then finish; else $\mathcal{P}_{fin} \leftarrow \mathcal{P}_{fin} \cup \mathcal{P}_{worst}$ and go to 1.

In light of the methods surveyed in [45], this procedure belongs to the category of "exchange methods" to solve semi-infinite optimization problems. These methods iteratively update the finite set $\mathcal{P}_{fin}$ by adding new worst-case points. It could also be conceivable to delete occasionally points from $\mathcal{P}_{fin}$. The selection of the initial $\mathcal{P}_{fin}$ at Step 0 can be done in several ways. For instance, open-loop simulations can be performed to select extreme cases on basis of step responses. Since the set $\mathcal{P}_{fin}$ is updated iteratively, in most cases it is enough to choose a single parameter set (e.g. the nominal one) to start the iterations. Note that according to [5, 46], the parameter combinations in $\mathcal{P}_{fin}$ define *multi-models* which allow an efficient evaluation of criteria based on linearized models. The result of the optimization at Step 1, T^*, defines a controller for the FCS, which is possibly not feasible in all points of $\mathcal{P}$. The purpose of the parameter study at Step 2 is to select efficiently new parameter combinations which are not covered by the existing ones in $\mathcal{P}_{fin}$. This time, simulations of the closed-loop system can serve to select worst-case parameter combinations. Alternatively, optimization driven worst-case search procedures can be used. The optimization procedure can be finished if no constraint violations occur. To cope with higher accuracy requirements, a grid refinement can be added to step 3, followed by resuming Step 2 on the finer grid.

We present shortly some of results obtained in [5] for the design of a robust longitudinal autopilot for the automatic landing of a modern cargo aircraft. This problem has been formulated as a part of the robust control benchmark problem for RCAM [3]. For controller design the used 12 tuners were the diagonal entries in the state and control weighting matrices defining a state-feedback design using the LQG-methodology. For controller tuning, 18 design criteria have been defined to meet the formulated design specifications along a prescribed landing trajectory. As uncertain parameters in the RCAM, the mass and two coordinates of the center of gravity have been used. Three parameter combinations have been selected to define $\mathcal{P}_{fin}$ with help of open-loop simulations: a nominal one and two "worst-cases" selected on basis of extreme differences in the shapes of step responses. This led to a multi-criteria optimization problem with $3 \times 18 = 54$ criteria. It is a remarkable result of [5] that with help of only three parameter combinations, all criteria have been met over the whole range of allowed values of the uncertain parameters. Moreover, post-design assessment results [6] showed that the multi-criteria based design of [5] had the best robustness properties among the 12 designs reported in [2].

In summary, the main computational challenge in optimization based robust FCS design is the efficient solution of semi-infinite multi-criteria optimization problems. Heuristic approaches exploiting particular problem features can help to reduce drastically the computational complexity. A concrete FCS-design [5] performed for the GARTEUR Robust Flight Control Design Challenge illustrated the applicability of such an approach.

4. Computational Challenges in the Assessment of Flight Controllers

While in the design of control systems satisfactory suboptimal solutions (corresponding to local minima) are generally acceptable, for the flight controller assessment, critical performance criteria like stability must be fulfilled over the whole flight envelope and for any possible combination of uncertain parameters. Thus, the performance robustness assessment of designed controllers must be formulated as a global optimization problem of the form

$$\begin{array}{ll} \text{maximize} & c(p) \\ p \in \mathcal{P} & \end{array} \tag{16}$$

where the maximization of the performance specification $c(p)$ is equivalent to an optimization driven search for the "worst-case" parameter combination in $\mathcal{P}$, a compact multidimensional interval of $\mathbb{R}^q$. The controller robustness assessment problem can be formulated simultaneously for several performance measures as well.

The most straightforward way to perform robustness assessment is to evaluate the function $c(p)$ on a grid of values in $\mathcal{P}$. However, the computation time required goes exponentially with the dimension. For $q = 4$, 16 and 64, the time needed to check only the 2^q vertices on a typical workstation is 0.01 seconds, 40 seconds, and 3.6×10^8 years, respectively. Thus, evaluating any robustness measure is NP hard in the number of parameters and obviously the computational costs for large problems are too high to be afordable in industrial practice.

There are several approaches to overcoming this apparent intractability. Monte Carlo simulation has been the industry standard for decades as an indirect approach for robustness analysis. The main advantage of Monte Carlo approach lies in that it determines "soft" bounds on hypothesis test whose accuracy does not depend on the dimension of the parameter space. The only involved cost is that of evaluating $c(p)$ a number of times to get statistically significant sample size. The main difficulty with the Monte Carlo soft bounds approach is it doesn't actually compute the probability distribution of the performance, but only indirectly assess it. This is especially a serious problem in FCS assessment, where we simply want to know if anything bad can happen for some set of parameters and not only to have a probability that a controller has acceptable performance for, say 99% of the uncertainty set $\mathcal{P}$, as has been observed on basis of 95% of the performed experiments. For a pertinent discussion of the above aspects and possible approaches to compute "hard" bounds on probability measures see [48].

In what follows we discuss how to compute hard bounds on performance measures and how to refine the bounds by using *branch and bound* (B&B) technique. This technique is particularly useful for performance criteria for which easily computable lower and upper bounds exist. The simplest B&B scheme is to choose a branching variable and splitting the corresponding interval in two equal parts, thus creating two new independent problems on which bounds are computed. A branch can be pruned when its local upper bound is lower than at least one of local lower bounds in other branches. Note that without pruning, the computational complexity of branching is exponential. More sophisticated algorithms would optimize both the chosen variable as well the location of the interval cut.

As already mentioned, the B&B approach is generally applicable if lower and upper bounds can be easily computed. Well suited for using B&B is the stability robustness measure expressed by the structured singular value μ [49, 50]. Although the computation of μ is an NP hard problem, upper and lower bounds can be computed in polynomial times. Another application area is when the criterion to be minimized can be expressed only as elementary operations with uncertain variables. In this case interval computations can be used to determine appropriate bounds for using the B&B approach.

Post-design stability robustness results for the RCAM benchmark problem are reported in [6]. The computations have been performed using the RCAM LFT-based uncertainty model [7] connected in feedback configurations with all of 12 designed controllers (see [2]). The same uncertain parameters have been used as for the controller design, that is, the mass and two coordinates of the center of gravity. The μ-analysis provided for RCAM the true magnitudes of the worst-case destabilizing parameter combinations, so that no B&B refining was necessary. Values of μ greater than 1 resulted only for three of the 12 controllers. In each case, it was possible to determine the corresponding worst parameter combination leading to instability. This combination was always lying in a vertex of the parameter space and nonlinear simulations confirmed the loss of stability in each case.

Since lower bounds are always readily available (for instance by choosing at random a parameter value, or better, performing a local maximization), the applicability of B&B is usually restricted by the lack of easy to compute approximations for the upper bound. Therefore, the B&B method is not applicable for more general FCS design criteria, whose evaluations involve possibly nonlinear simulation, trimming and eigenvalue computation, or frequency response calculation. For such criteria, more general global optimization methods have to be used.

The simplest approach for general criteria is to use a local minimizer as local worst-case search engine, performing optimizations initialized in many different points of the parameter space. This approach has been successfully used for an alternative stability robustness analysis performed directly on the nonlinear closed-loop aircraft model. This model has been built by coupling the nonlinear aircraft model (1) with each of the designed 12 controllers for RCAM. As a measure of the stability degree, the minimum damping of the eigenvalues has been used. A single evaluation of the minimum damping for a given parameter value p and a specified flight-condition involves trimming and linearizing of the nonlinear model in the found equilibrium point, followed by the computation of the eigenvalues of the closed-loop state dynamics matrix. The local optimization has been performed repeatedly from different initial points, trying to destabilize the closed-loop system by minimizing the minimal damping within the operating envelope over all uncertainty-parameter combinations. The obtained results agree well with those obtained with the μ computation.

Although usually feasible, the local search approach can not provide a 100% guarantee for assessment results. Obviously, the only way to obtain such a guarantee is to use global optimization algorithms like genetic algorithms (GA) [51] or simulated annealing (SA) [52] for performance robustness analysis. The study of these algorithms in the context of stability robustness analysis [49] demonstrated that both methods are easy to use and can provide reasonably good answers for that particular problem. The comparison with the B&B approach shows that GA and SA are not competitive with the B&B method with respect to computational efficiency. Still, GA and SA are the only methods which can be used in a more general computation like that discussed in the previous paragraph.

In summary, the computational challenges for the assessment of performance robustness of flight controllers have particular features depending on the type of global optimization problems to be solved:

- for the computation of hard bounds in special cases (e.g. assessment of stability robustness), the B&B technique is the main favorite; here it would be interesting to extend the applicability of this technique for the cases when no easy evaluation of upper bounds is possible
- for global optimization based assessment of general robust flight control design criteria, both GA and SA can be used, but the cost to use these methods could be very high for many practical applications; here also, a combination with the more efficient B&B technique would be desirable.

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